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ON INFINITE PRODUCTS OF FUNCTIONS IN COMPACT RIEMANN SURFACES

BY

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ABSTRACT

Let $\mathcal{V}'$ be the complementary of a point ∞ in a compact Riemann surface $\mathcal{V}$. The normal convergence in compact subsets of $\mathcal{V}'$ of an infinite product of meromorphic functions (with polynomic exponential singularities at ∞ of bounded degree) is shown in this paper to be equivalent to a certain type of convergence of the double series of Newton sums of the divisors of its factors. This applies, for instance, to products of Baker functions in $\mathcal{V}'$ and to products of meromorphic functions in $\mathcal{V}$. The result for this last case is also generalized to complementaries of arbitrary nonvoid finite subsets of $\mathcal{V}$.

1. Introduction

Let $\mathcal{V}$ be a compact (connected) Riemann surface of genus g, ∞ be a non-Weierstrass point of $\mathcal{V}$, and $\mathcal{V}' = \mathcal{V} - \{\infty\}$. Let z be a coordinate in some open neighbourhood V of ∞ such that $z(\infty) = 0$. For every $j \in \mathbb{N}$ and every divisor δ in $\mathcal{V}$ supported in V, that is $\delta = \sum_{i=1}^{r} n_i a_i$, with $n_i \in \mathbb{Z}$ and $a_i \in V$ for $i = 1, \ldots, r$, let $\sigma_j(\delta)$ be $\sum_{i=1}^{r} n_i z(a_i)^j$. We shall call $\sigma_j(\delta)$ the j-th Newton sum of the divisor δ with respect to the coordinate z.

Let G_∞ be the multiplicative group of the Baker functions in $\mathcal{V}'$, i.e., those meromorphic functions in $\mathcal{V}'$ with polynomic exponential singularity at ∞ of degree $\leq g$ (by which we mean that they are in $V - \{\infty\}$ of the form $he^{P(1/z)}$, with h meromorphic in V and P a polynomial of degree $\leq g$). Let (f_n) be a

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sequence in G_∞ such that both of its sequences (δ_n^+) of zero divisors and (δ_n^-) of polar divisors have bounded degree $\leq k \in \mathbb{N}$. Assume also that (f_n) verifies the following two conditions (which are obviously fulfilled if the infinite product $\prod_{n=1}^{\infty} f_n$ converges uniformly in the compact subsets of $\mathcal{V}'$):

(1) The sequence $(\delta_n) = (\delta_n^+ - \delta_n^-)$ tends to ∞ (in the obvious sense that every neighbourhood of ∞ contains all but finitely many supports of these divisors).

(2) There is some point $q_0 \in \mathcal{V}'$ such that $\prod f_n(q_0)$ converges, the product extending over the f_n without pole at q_0.

Then, it was proved in [3] that the absolute convergence of the series (of the defined terms in) $\sum_{n=1}^{\infty} \sigma_j(\delta_n)$, for $1 \leq j \leq k$, implies that $\prod_{n=1}^{\infty} f_n$ converges normally in the compact subsets of $\mathcal{V}'$, the two properties being equivalent if the functions f_n are meromorphic in $\mathcal{V}$ (in fact, condition (2) above was replaced in [3] by the hypothesis that the f_n without pole at q_0 are normalized so that they take the value 1 at this point).

Suppose now that (δ_n) is a sequence of arbitrary finite divisors in $\mathcal{V}'$ tending to ∞, and let (f_n) be a sequence of normalized (in the above sense) Baker functions in $\mathcal{V}'$ having (δ_n) as sequence of divisors (see, for instance, [2]). In this general situation (without requiring that the corresponding sequences (δ_n^+) of zero divisors and (δ_n^-) of polar divisors have bounded degree), a natural question is if some analogue of the above result holds. For instance, does the convergence of $\sum_{n=1}^{\infty} |\sigma_j(\delta_n)|$, for every $j \in \mathbb{N}$, imply the convergence of $\prod_{n=1}^{\infty} f_n$?

Our purpose in this paper is to study the existence of sequences (f_n) of meromorphic functions in $\mathcal{V}'$, with polynomic exponential singularities at ∞ of bounded degree, having a prescribed sequence (δ_n) of finite divisors (tending to ∞), and such that $\prod_{n=1}^{\infty} f_n$ converges uniformly in the compact subsets of $\mathcal{V}'$. We shall in fact show, among other results, that the above one does have a generalization, but also that the mentioned hypothesis on the absolute convergence of every series of j-th Newton sums is not sufficient, there being necessary another condition on the double series $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$ to guarantee the convergence of $\prod_{n=1}^{\infty} f_n$.

In Section 2 we shall introduce some terminology and expose some results to be used later. Part of them will be used to improve the cited result of [3] (in the particular case of meromorphic functions in $\mathcal{V}$) by showing that the hypothesis of the absolute convergence of the first g series of Newton sums is superfluous. The general case of products of functions with arbitrary finite divisors tending to ∞ will be treated in Section 3, in which we introduce the concept of G-convergence of a sequence of such divisors, and use it to characterize the products which are normally convergent in the compact subsets of $\mathcal{V}'$. In Section 4 we shall deal with the existence of sequences of positive divisors with certain properties (the term "positive" being used in this paper to emphasize that we refer to effective divisors with at least one nonzero coefficient). Finally, in the last section we shall briefly generalize some results to the case in which ∞ may be of Weierstrass, and also to the case of $\mathcal{V}'$ being the complementary of any nonvoid finite subset of $\mathcal{V}$.

Besides the above explained notation, we shall also use the following:

For every open subset U of $\mathcal{V}$, $O(U)$ and $M(U)$ will respectively be the rings of holomorphic functions in U and of meromorphic functions in U. $G(\mathcal{V}')$ will be the multiplicative group of the functions with finite divisor in $\mathcal{V}'$.

For simplicity, the genus g of $\mathcal{V}$ will be supposed to be > 0, although the results whose statements make sense in the case $g = 0$ are also true in this case. We shall consider piecewise C^1 curves in $\mathcal{V}'$, $A_1, \ldots, A_g, B_1, \ldots, B_g$, defining a canonical system of generators for the fundamental group of $\mathcal{V}$ (following, for instance, the terminology in [5]), and Δ will be the simply connected open subset of $\mathcal{V}$ complementary of the union of these curves.

$\omega_1, \ldots, \omega_g$ will be a basis of the space of holomorphic differentials in $\mathcal{V}$ verifying $\int_{A_i} \omega_j = \delta_{ij}$ (Kronecker's delta). For $j = 1, \ldots, g$, θ_j will be the unique holomorphic differential in $\mathcal{V}'$ such that $\theta_j - dz/z^{j+1}$ is holomorphic in V and such that $\int_{A_\ell} \theta_j = 0$, for $\ell = 1, \ldots, g$. For every pair a, b of points in Δ, θ_{ab} will be the unique holomorphic differential in $\mathcal{V} - \{a, b\}$ having simple poles at a, b with residues $1, -1$, respectively, and such that $\int_{A_\ell} \theta_{ab} = 0$, for $\ell = 1, \ldots, g$. For every divisor $\delta = \sum_{i=1}^{r} n_i a_i$, with $n_i \in \mathbb{Z}$ and $a_i \in \mathcal{V}$, $\theta_{\delta\infty}$ will briefly denote the sum $\sum n_i \theta_{a_i\infty}$, extended over the a_i different from ∞.

For every function ψ holomorphic in some open neighbourhood W of ∞, we shall also represent by ψ its natural extension to finite divisors $\delta = \sum_{i=1}^{r} n_i a_i$ supported in W, i.e., $\psi(\delta) = \sum_{i=1}^{r} n_i \psi(a_i)$. Note that if $\psi = z^j$ for some $j \in \mathbb{N}$, then $\psi(\delta)$ is the j-th Newton sum of δ with respect to z.

From now on we shall suppose that V is a coordinate disk of radius 1 *with respect to z, centered at ∞* (i.e., $z(V)$ is the open disk in $\mathbb{C}$ of radius 1 centered at $z(\infty) = 0$). We shall also assume without loss of generality that $\overline{V}$ is contained in Δ. V' will be $V - \{\infty\}$. A V-disk will be a coordinate disk D with respect to z, centered at ∞, and such that $\overline{D} \subset V$. Unless otherwise stated all considered Newton sums are with respect to z.

The following terminology, part of which has been already used, will be utilized too:

Let $q_0 \in \mathcal{V} - \overline{V}$, fixed from now on. We shall say that a function $f \in M(\mathcal{V}')$, having no zero or pole at the point q_0, is normalized, if $f(q_0) = 1$. If f has a zero or a pole at q_0, we require no condition on f to be normalized, i.e., every such f is normalized. A sequence (f_n) in $M(\mathcal{V}')$ will be said to be normalized if all the f_n are normalized.

We shall say for short that an infinite product of meromorphic functions in an open subset U of $\mathcal{V}$ converges normally, if it converges normally in all compact subsets of U. The analogous shortening will be used for series of functions or differentials.

The following proposition appearing in [3] will be useful.

Proposition 1.1: *For every $a \in \Delta - \{\infty\}$ and $j = 1, \dots, g$, let $\varphi_j(a) \in \mathbb{C}$ be such that $\theta_{a\infty} + \sum_{j=1}^{g} \varphi_j(a)\theta_j$ has null integrals along $B_1, \dots, B_g$. Then, $\varphi_1, \dots, \varphi_g$ prolong to holomorphic functions in Δ, and each sum $\varphi_j + z^j$, with $1 \le j \le g$, has a zero of order $\ge g+1$ at ∞.*

From now on, whenever we consider a series having possibly a finite number of undefined terms (for instance, $\sum_{n=1}^{\infty} \sigma_j(\delta_n)$, with $j \in \mathbb{N}$, and (δ_n) being a sequence of divisors in $\mathcal{V}$ tending to ∞) and we say that it converges in some sense, it must be understood that by getting rid of those terms the resulting series converges in the indicated way. The same will be valid for infinite products.

Every sequence of divisors in $\mathcal{V}$ tending to ∞ will be supposed, if necessary, to have all its terms supported in V or even in some convenient V-disk (or neighbourhood of ∞). It will be always clear that this is correct and means no loss of generality.

2. Preliminaries and an improvement of a previous theorem

We shall recall in this section some results and explain some more useful terminology and notation. We shall also improve the result of [3] mentioned in the Introduction.

First, we introduce for brevity, the following:

Definition 2.1: A function $f \in M(\mathcal{V}')$ will be called Δ-simple, if its divisor is supported in Δ (i.e., all its zeros and poles are in Δ), and if the integrals of $d\log f$ along $A_1, \dots, A_g, B_1, \dots, B_g$ are null.

The following two propositions are elementary.

Proposition 2.2: *Let $(a_n), (b_n)$ be two sequences in a nonvoid open subset U of $\mathbb{C}^k$ converging to a same point of U. If $\sum_{n=1}^{\infty} \|b_n - a_n\|$ converges (for* *some norm in $\mathbb{C}^k$), then $\sum_{n=1}^{\infty} F(b_n) - F(a_n)$ converges absolutely for every holomorphic function F in U.*

PROPOSITION 2.3: *If $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, with $f_n \in M(\mathcal{V}')$ for every $n \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that f_n is Δ-simple for every $n \geq n_0$.*

We now put together some properties of Δ-simple functions.

PROPOSITION 2.4: (1) *If $h \in M(\mathcal{V})$ and its divisor δ is supported in Δ, then $d\log h = \theta_{\delta\infty} + \sum_{j=1}^{g}(\int_{A_j} d\log h)\omega_j$. In particular, if h is Δ-simple, then $d\log h = \theta_{\delta\infty}$.*

(2) *If $f \in G_\infty$ is Δ-simple and its divisor is δ, then*

$$d\log f = \theta_{\delta\infty} + \sum_{j=1}^{g} \varphi_j(\delta)\theta_j.$$

(3) *A divisor δ in $\mathcal{V}$, supported in Δ, is the divisor of a Δ-simple function of $M(\mathcal{V})$ if and only if $\varphi_j(\delta) = 0$ for $1 \leq j \leq g$.*

Proof: (1) $d\log h - (\theta_{\delta\infty} + \sum_{j=1}^{g}(\int_{A_j} d\log h)\omega_j)$ is a holomorphic differential in $\mathcal{V}$ with null integrals along $A_1, \ldots, A_g$.

(2) $d\log f - (\theta_{\delta\infty} + \sum_{j=1}^{g} \varphi_j(\delta)\theta_j)$ is a holomorphic differential in $\mathcal{V}'$ with a pole of order $\leq g+1$ at ∞. Since it has null integrals along $A_1, \ldots, A_g, B_1, \ldots, B_g$, it is the logarithmic differential of some holomorphic function in $\mathcal{V}'$ with a pole of order $\leq g$ at ∞, which must be a constant.

(3) Every divisor δ in $\mathcal{V}$, supported in Δ, is the divisor of a Δ-simple function of G_∞ (see, for instance, [2]). By the definition of the φ_j and (2), this function is meromorphic in $\mathcal{V}$ if and only if $\varphi_j(\delta) = 0$, for $1 \leq j \leq g$. ∎

The next result is not explicitly stated in [3], but is in fact proved in that paper. It is a consequence of a theorem in Royden [9], which can be proved as in (1)⇒(2) in Theorem 2.12 of [3].

THEOREM 2.5: *Let (f_n) be a normalized sequence of Δ-simple functions in $G(\mathcal{V}')$ such that its sequence of divisors tends to ∞. Then, $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$ if and only if for every V-disk D and every holomorphic function ψ in some neighbourhood of $\overline{D}$ taking the value 0 at ∞, the series $\sum_{n=1}^{\infty} \int_{\partial D} \psi \, d\log f_n$ converges absolutely.*

Recall now that the set of positive divisors of degree k supported in any V-disk D can be considered as the k-th symmetric product $D^{(k)}$, and can be naturally endowed with a k-dimensional complex manifold structure, it being well known that in this manifold the Newton sums are coordinates, i.e., the mapping from $D^{(k)}$ into $\mathbb{C}^k$ with components $\sigma_1, \ldots, \sigma_k$ is a holomorphic isomorphism of $D^{(k)}$ with an open subset of $\mathbb{C}^k$ ([4], [5], [8]). The functions $\lambda_1, \ldots, \lambda_k$ defined in $V^{(k)}$ by the equality of polynomials in z, $z^k + \lambda_1(\delta)z^{k-1} + \cdots + \lambda_{k-1}(\delta)z + \lambda_k(\delta) = (z-z(p_1)) \cdots (z-z(p_k))$, for every $\delta = p_1 + \cdots + p_k \in V^{(k)}$, also form a coordinate system in $V^{(k)}$. If we further define $\lambda_j(\delta)$ to be 0 for $j \geq k+1$, then between both sets of functions $\{\sigma_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}}$, there exist the relations expressed by the following Newton formula, valid for every $j \in \mathbb{N}$ and every $\delta \in V^{(k)}$ (see, for instance, Chrystal [1]):

$$\sigma_j(\delta) + \lambda_1(\delta)\sigma_{j-1}(\delta) + \cdots + \lambda_{j-1}(\delta)\sigma_1(\delta) + j\lambda_j(\delta) = 0. \tag{2.5.1}$$

Remark: It is easy to deduce from (2.5.1), by induction on j, that if $|\sigma_j(\delta)| < R^j$ for some $R > 0$ and $1 \leq j \leq k$, then $|\lambda_j(\delta)| < R^j$ also holds for $1 \leq j \leq k$. Similarly, one proves that $|\lambda_j(\delta)| < R^j$, for $1 \leq j \leq k$, implies that $|\sigma_j(\delta)| < (2^j - 1)R^j$, for $1 \leq j \leq k$.

From now on, in this section, (h_n) will be a normalized sequence of functions in $M(\mathcal{V})$ with degrees bounded by $k \in \mathbb{N}$, and (δ_n) will be the corresponding sequence of divisors. Since we have supposed that ∞ is not a Weierstrass point, condition (1) of the Introduction for the sequence (h_n) does not make sense if $k \leq g$. Therefore, we shall restrict ourselves to the case $k \geq g+1$.

The following analogue of Abel's theorem implies that functions in $M(\mathcal{V})$ with degree $\leq k$ and zeroes and poles sufficiently close to ∞ must be Δ-simple.

THEOREM 2.6: *There exists a V-disk D (depending on k) such that a divisor $\delta = \delta^+ - \delta^-$, with $\delta^+, \delta^- \in D^{(k)}$, is principal if and only if $\varphi_j(\delta) = 0$, for $1 \leq j \leq g$.*

Proof: Integrate both members of the first equality in (1) of Proposition 2.4 along the curves $B_1, \ldots, B_g$, and reason for instance as in the proof of Theorem 2.17 in [3]. ∎

Despite the similarity of Theorem 2.6 with (3) of Proposition 2.4, note the difference between both statements. In fact, an arbitrary function $h \in M(\mathcal{V})$ with divisor supported in Δ need not be Δ-simple, this condition being equivalent to $\varphi_j(\delta) = 0$, for $1 \leq j \leq g$, where δ is the divisor of h.

The coordinate system considered in the following theorem will be essential to obtain the improvement mentioned in the Introduction.

THEOREM 2.7: *The functions $\varphi_1, \ldots, \varphi_g, \sigma_{g+1}, \ldots, \sigma_k$ form a coordinate system in $D^{(k)}$, for some V-disk D.*

Proof: It is a consequence of Remark 1 of III.11.9 in [4] and Proposition 1.1 (or part "a" of the Proposition in III.11.11 of [4], applied to the divisor $g\infty$, bearing in mind the relations between $\varphi_1, \ldots, \varphi_g$ and the functions $\phi_j(a) = \int_\infty^a \omega_j$). ∎

COROLLARY 2.8: *If $\sum_{n=1}^\infty |\sigma_j(\delta_n)|$ converges for $g+1 \le j \le k$, then $\sum_{n=1}^\infty |\psi(\delta_n)|$ also converges for every holomorhic function ψ in some neighbourhood of ∞ such that $\psi(\infty) = 0$.*

Proof: By Proposition 2.2 and Theorems 2.6 and 2.7. ∎

THEOREM 2.9: *$\prod_{n=1}^\infty h_n$ converges normally in $\mathcal{V}'$ if and only if (δ_n) tends to ∞ and $\sum_{n=1}^\infty \sigma_j(\delta_n)$ converges absolutely for $g+1 \le j \le k$. Furthermore, any one of these equivalent conditions implies the absolute convergence of $\sum_{n=1}^\infty \sigma_1(\delta_n), \ldots, \sum_{n=1}^\infty \sigma_g(\delta_n)$.*

Proof: Let D be a V-disk, and suppose that $\prod_{n=1}^\infty h_n$ converges normally in $\mathcal{V}'$. Then, multiplying by z^j the series $\sum_{n=1}^\infty d\log h_n$, integrating along the boundary of D, and using (1) of Proposition 2.4, one easily arrives at the conclusion that $\sum_{n=1}^\infty |\sigma_j(\delta_n)|$ converges for $j = 1, \ldots, k$.

Conversely, assume that these series converge for $g+1 \le j \le k$ and that (δ_n) tends to ∞. Let ψ be a holomorphic function in some neighbourhood of $\overline{D}$ verifying $\psi(\infty) = 0$. Then, again by (1) of Proposition 2.4 we have $\int_{\partial D} \psi \, d\log h_n = 2\pi i \psi(\delta_n)$ for every $n \in \mathbb{N}$. Therefore, by Corollary 2.8, $\int_{\partial D} \psi \, d\log h_n$ converges absolutely, from which by Theorem 2.5 one deduces that $\prod_{n=1}^\infty h_n$ converges normally in $\mathcal{V}'$. ∎

Apologies for an error (of minor importance) in [3]: J. M. Verde and the author apologize to the editors and to the readers of Israel J. Math. for their insufficient care while thinking through some details of the proof of Theorem 2.16 in [3]. In fact, the first two of the five supposedly equivalent conditions appearing in that theorem are really equivalent and imply the convergence of the product there considered, but the other three conditions (which are also equivalent to each other) do not imply in general the first ones if $k \ge g+1$, as can be deduced for instance from Theorem 2.7 of this paper.

3. General case

First, we introduce a certain class of double series of complex numbers and expose some of their properties. We shall explain later the relation between this kind of series and normally convergent products of functions.

Definition 3.1: Given $p \in \mathbb{N}$, a double series $\sum_{j,n=1}^{\infty} \alpha_{j,n}$, with $\alpha_{j,n} \in \mathbb{C}$ for all $j, n \in \mathbb{N}$, will be called geometrically convergent from the p-th row (G_p-convergent for short) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\alpha_{j,n}| < \epsilon^j$ for every $j \geq p$.

Note that, for every $N \in \mathbb{N}$, $\sum_{j,n=1}^{\infty} \alpha_{j,n}$ is G_p-convergent if and only if the same is true for $\sum_{j \in \mathbb{N}, n \geq N} \alpha_{j,n}$. Observe also that, assuming the convergence of $\sum_{n=1}^{\infty} |\alpha_{j,n}|$ for $j \geq p$, it suffices that the inequality of the definition holds for all sufficiently large values of j in order that it does (with a possibly different n_0) for all $j \geq p$.

If necessary in the sequel (because of not being explicitly mentioned), whenever we consider a double series, it will be supposed to have complex terms. We state now some clear consequences of the definition.

PROPOSITION 3.2: *If $p \in \mathbb{N}$ and $\sum_{j,n=1}^{\infty} \alpha_{j,n}$ is G_p-convergent, we have:*

(1) *$\sum_{j \geq p, n \geq n_0} |\alpha_{j,n}|$, with n_0 as in Definition 3.1, converges (note, however, that this may not be true for $\sum_{j \geq p, n \geq 1} |\alpha_{j,n}|$).*

(2) *The sum of $\sum_{j,n=1}^{\infty} \alpha_{j,n}$ with any other G_p-convergent double series has this property too.*

(3) *For every $R > 0$, $\sum_{j,n=1}^{\infty} \alpha_{j,n} R^j$ is also G_p-convergent.*

Another easy consequence of the definitions is the following:

PROPOSITION 3.3: *For every $p \in \mathbb{N}$ and $\sum_{j,n=1}^{\infty} \alpha_{j,n}$, such that $\sum_{n=1}^{\infty} \alpha_{j,n}$ is absolutely convergent for $j \geq p$, the following conditions are equivalent:*

(1) *$\sum_{j,n=1}^{\infty} \alpha_{j,n}$ is G_p-convergent.*

(2) $\lim_{j,\ell \to \infty} \sqrt[j]{\sum_{n=\ell}^{\infty} |\alpha_{j,n}|} = 0$.

(3) *For every $\epsilon > 0$ there exist $n_0, m \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\alpha_{j,n}| < \epsilon^{[j/m]}$, for all $j \geq p$ (or sufficiently large), where $[j/m]$ is the integer part of j/m.*

We now begin to explain the relation of this type of series with infinite products of functions. Firstly, we prove an auxiliary result in whose statement $\mathbb{D}$ is the unit disk in $\mathbb{C}$ (i.e., the open disk with radius 1 centered at 0), $\mathcal{D}$ is the group of finite divisors in $\mathbb{D}$, and $s_j(\delta)$ is the j-th Newton sum of δ with respect to the natural coordinate in $\mathbb{C}$ (defined in the same way as in the Introduction) for every $\delta \in \mathcal{D}$ and $j \in \mathbb{N}$.

LEMMA 3.4: *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic and univalent, with $f(0) = 0$, and let $\tilde{f}\colon \mathcal{D} \to \mathcal{D}$ be the unique group homomorphism which, restricted to $\mathbb{D}$ (identified with the subset of $\mathcal{D}$ formed by the positive divisors of degree 1), coincides with f. Then, one has $|s_j(\tilde{f}(\delta))| \leq \sum_{m=j}^{\infty} |s_m(\delta)|$, for every $\delta \in \mathcal{D}$ and $j \in \mathbb{N}$.*

Proof: Let $\sum_{m=j}^{\infty} \mu_{j,m} z^m$ be the power series expansion of f^j in $\mathbb{D}$, and consider some $\delta = a_1 + \cdots + a_r - (b_1 + \cdots + b_s) \in \mathcal{D}$. Then,

$$\begin{aligned}
&|s_j(\tilde{f}(\delta))| \\
&\quad = |f(a_1)^j + \cdots + f(a_r)^j - (f(b_1)^j + \cdots + f(b_s)^j)| \\
&\quad = \left| \sum_{m=j}^{\infty} \mu_{j,m} a_1^m + \cdots + \sum_{m=j}^{\infty} \mu_{j,m} a_r^m - \left(\sum_{m=j}^{\infty} \mu_{j,m} b_1^m + \cdots + \sum_{m=j}^{\infty} \mu_{j,m} b_s^m \right) \right| \\
&\quad = \left| \sum_{m=j}^{\infty} \mu_{j,m} s_m(\delta) \right|.
\end{aligned}$$

Since $\sum_{m=j}^{\infty} m|\mu_{j,m}|^2 \leq j$ (the well-known case $j = 1$ and $f'(0) = 1$ can be found for instance in [7], it being easy to see that the arguments there used for this particular case can be adapted to prove the general one), it is clear that one deduces the inequality of the statement. ∎

We shall see later that for $p \geq g + 1$, the following theorem is an indirect consequence of the proofs of Theorems 3.11 and 3.14 below. Nevertheless, we have included Lemma 3.4 and the brief proof of the theorem for the sake of completeness and logic of the exposition.

THEOREM 3.5: *Let (δ_n) be a sequence of divisors in $\mathcal{V}$ tending to ∞. Then, for every $p \in \mathbb{N}$, the condition that $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$ is G_p-convergent is independent of the coordinate z considered (verifying $z(\infty) = 0$).*

Proof: Let w be another coordinate in a neighbourhood of $\overline{V}$ (reduce V if necessary) such that $w(\infty) = 0$. By (3) in Proposition 3.2, we can suppose without loss of generality that $|w| \leq |z|$ in V. For every finite divisor δ in V and $j \in \mathbb{N}$, let $\sigma'_j(\delta)$ be the j-th Newton sum of δ with respect to w. Then from Lemma 3.5, applied to $f = w \circ z^{-1}$, one deduces that for all $j, n \in \mathbb{N}$, we have $|\sigma'_j(\delta_n)| \leq \sum_{m=j}^{\infty} |\sigma_m(\delta_n)|$.

Assume now that $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$ is G_p-convergent. Then, for every $\epsilon \in (0, 1/2)$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\sigma_m(\delta_n)| < \epsilon^m$ for $m \geq p$. So, for

$j \geq p$, $\sum_{n=n_0}^{\infty} |\sigma'_j(\delta_n)| \leq \sum_{n=n_0}^{\infty} \sum_{m=j}^{\infty} |\sigma_m(\delta_n)| < \sum_{m=j}^{\infty} \epsilon^m = \epsilon^j/(1-\epsilon)$, from which one easily deduces that $\sum_{j,n=1}^{\infty} \sigma'_j(\delta_n)$ is also G_p-convergent. ∎

We want now to use Theorem 3.5 to define a certain type of convergence of sequences of divisors in $\mathcal{V}$, but we first justify the use of the term "convergence" with the next theorem, which also shows that in Theorem 3.5 the hypothesis that the sequence (δ_n) tends to ∞ is essentially superfluous and can be replaced by the assumption (in any case necessary in order that the $\sigma_j(\delta_n)$ are defined) that almost all the δ_n are supported in V (or any other coordinate disk containing ∞). Its statement will be preceded by a lemma in which we again use the notations of Lemma 3.4.

LEMMA 3.6: *Let δ be a divisor in $\mathbb{C}$ of the form $\delta_1 + \delta_2$, where $\delta_1 = \sum_{i=1}^{r} n_i a_i$, with $a_i \in \partial\mathbb{D}$ all different and $n_i \in \mathbb{Z}^*$, and $\delta_2 \in \mathcal{D}$. Then there is a subsequence $(s_{j_k}(\delta))$ of $(s_j(\delta))$ such that $|s_{j_k}(\delta)| > 1/2^{j_k}$ for every $k \in \mathbb{N}$.*

Proof: As is well known, the $s_j(\delta_1)$ are the coefficients of the Taylor series in some disk centered at 0 of F'/F, where F is a rational function in $\mathbb{C}$ having $\sum_{i=1}^{r} n_i a_i^{-1}$ as divisor. We deduce that the convergence radius of this series is 1 and so $\overline{\lim}_{j\to\infty} \sqrt[j]{|s_j(\delta_1)|} = 1$. Similarly, we see that $\overline{\lim}_{j\to\infty} \sqrt[j]{|s_j(\delta_2)|} < 1$. Therefore, there is a sequence (j_k) of subindexes such that $\lim_{k\to\infty} \sqrt[j_k]{|s_{j_k}(\delta_1)|} = 1$, and such that $\lim_{k\to\infty}(s_{j_k}(\delta_2)/s_{j_k}(\delta_1)) = 0$. Since this implies that there exists $k_0 \in \mathbb{N}$ such that $|s_{j_k}(\delta_1)| > (2/3)^{j_k}$ and $|s_{j_k}(\delta)| > 3/4|s_{j_k}(\delta_1)|$, for $k \geq k_0$, we easily obtain the desired conclusion. ∎

THEOREM 3.7: *Let (δ_n) be a sequence of finite divisors in V. If $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$ is G_p-convergent for some $p \in \mathbb{N}$, then (δ_n) tends to ∞.*

Proof: Consider any $\epsilon \in (0, 1/2)$, and let $n_0 \in \mathbb{N}$ be such that $\sum_{n=n_0}^{\infty} |\sigma_j(\delta_n)| < \epsilon^j$ for $j \geq p$. In particular, $|\sigma_j(\delta_n)| < \epsilon^j$ holds for every $n \geq n_0$ and $j \geq p$.

Let us now denote, for simplicity, by δ any of the δ_n with $n \geq n_0$. Then, one easily deduces from Lemma 3.6 that there exists a sequence of subindexes (j_k) such that $|\sigma_{j_k}(\delta)| > (R/2)^{j_k}$ for every $k \in \mathbb{N}$, where R is the maximum of the moduli of the coordinates of the points in the support of δ. Therefore, $R < 2\epsilon$, that is δ is supported in the V-disk with radius 2ϵ. ∎

Theorems 3.5 and 3.7, which of course can be applied to any other point q of $\mathcal{V}$, instead of ∞, and sequences of divisors in $\mathcal{V}$ tending to q (this will be useful in Section 5), imply that the following definition is correct.

Definition 3.8: Given $p \in \mathbb{N}$, we shall say that a sequence (δ_n) of divisors in $\mathcal{V}$ is G_p-convergent (or G-convergent, if $p = 1$) to a point $q \in \mathcal{V}$, if there exists a coordinate disk W containing q and all but a finite set of supports of the δ_n and if the double series $\sum_{j\in\mathbb{N},n\geq N}\sigma'_j(\delta_n)$ is G_p-convergent for some $N \in \mathbb{N}$, where $\sigma'_j(\delta_n)$ is the j-th Newton sum of δ_n with respect to a coordinate w in W such that $w(q) = 0$.

A sequence (δ_n) of divisors in $\mathcal{V}$, which is G_p-convergent to ∞, will be simply said to be G_p-convergent (or G-convergent, if $p = 1$).

Note that the G_p-convergence of the double series $\sum_{j\in\mathbb{N},n\geq N}\sigma'_j(\delta_n)$ of the definition does not depend on the $N \in \mathbb{N}$ considered (provided that $\sigma'_j(\delta_n)$ is defined for $n \geq N$).

Obvious consequences of the definition are:

(1) A sequence (δ_n) of divisors in $\mathcal{V}$ is G_p-convergent, for some $p \in \mathbb{N}$, if and only if $(\delta_n|_{\mathcal{V}'})$ is.

(2) If (δ_n) and (δ'_n) are G_p-convergent sequences of finite divisors in $\mathcal{V}$ for some $p \in \mathbb{N}$, then the same is true for $(\delta_n + \delta'_n)$.

(3) If $a_n \in V$ for every $n \in \mathbb{N}$ and $(b_n), (c_n)$ are sequences in V tending to ∞, and if the series $\sum_{n=1}^{\infty}|z(a_n)|, \sum_{n=1}^{\infty}|z(b_n) - z(c_n)|$ are convergent, then (a_n) and $(b_n - c_n)$ are G-convergent sequences of divisors in $\mathcal{V}$. If we only suppose that for some $p \in \mathbb{N}$, $\sum_{n=1}^{\infty}|z(a_n)|^p$ and $\sum_{n=1}^{\infty}|z(b_n) - z(c_n)|^p$ converge, then both (a_n) and $(b_n - c_n)$ are G_p-convergent.

Similarly as in (3) of above, but not as easily, we have:

Proposition 3.9: *Let (δ_n^+) and (δ_n^-) be two sequences in $V^{(k)}$ for some $k \in \mathbb{N}$, and δ_n be $\delta_n^+ - \delta_n^-$ for every $n \in \mathbb{N}$. Then, (δ_n) is G-convergent if and only if (δ_n) tends to ∞ and $\sum_{n=1}^{\infty}\sigma_j(\delta_n)$ converges absolutely for $1 \leq j \leq k$.*

Proof: One implication is clear from Theorem 3.7. Let us prove the other: Given $\epsilon > 0$, consider a V-disk W such that $|\lambda_i| < \epsilon/2k$ and $|\sigma_i| < 1$ in $W^{(k)}$ for $i = 1, \dots, k$ (notation as in 2.5.1). Let $n_0 \in \mathbb{N}$ be such that δ_n is supported in W for $n \geq n_0$, and such that $\sum_{n=n_0}^{\infty}|\sigma_i(\delta_n^+) - \sigma_i(\delta_n^-)| < 1$, $\sum_{n=n_0}^{\infty}|\lambda_i(\delta_n^+) - \lambda_i(\delta_n^-)| < \epsilon/2k$, for $i = 1, \dots, k$. Then, one easily deduces by induction from (2.5.1) that for every $j \geq k+1$, $|\sigma_j| < \epsilon^{[j/k]}$ in $W^{(k)}$. From this and from

$$\sigma_j(\delta_n^-)-\sigma_j(\delta_n^+) = \sum_{i=1}^{k}\lambda_i(\delta_n^+)(\sigma_{j-i}(\delta_n^+))-\sigma_{j-i}(\delta_n^-))+(\lambda_i(\delta_n^+)-\lambda_i(\delta_n^-))\sigma_{j-i}(\delta_n^-),$$

we can obtain, also by induction, that

$$\sum_{n=n_0}^{\infty} |\sigma_j(\delta_n^+) - \sigma_j(\delta_n^-)| = \sum_{n=n_0}^{\infty} |\sigma_j(\delta_n)| < \epsilon^{[j/k]}.$$

Hence, by (3) in Proposition 3.3, the desired conclusion follows. ∎

Remark: Note that if we replace in Proposition 3.9 the condition of the convergence of $\sum_{n=1}^{\infty} |\sigma_j(\delta_n)|$, for $1 \le j \le k$, by the weaker one of the convergence of these series for $p \le j \le k$, with $p \in \mathbb{N}$ and $2 \le p \le k$, one cannot obtain the conclusion that (δ_n) is G_p-convergent. To see this, consider any such p. Then, there exists a unique sequence (δ_n) in $V^{(k)}$ with $\sigma_j(\delta_n) = 1/(\log n)^j$ for $1 \le j \le p-1$ and every $n \in \mathbb{N}$ (and so, $\lambda_j(\delta_n) = 0$ for $2 \le j \le p-1$ and every $n \in \mathbb{N}$), and with $\sigma_j(\delta_n) = 0$ for $p \le j \le k$ and every $n \in \mathbb{N}$. Now, use (2.5.1) to see that

$$k \le 2p-2 \Rightarrow \sigma_j(\delta_n) + j\lambda_j(\delta_n) = \frac{-1}{(j-1)j(\log n)^j} \quad \text{for } p+1 \le j \le k+1,$$

and that $2p-1 \le k \Rightarrow \sigma_j(\delta_n) + j\lambda_j(\delta_n)$ is as above for $p+1 \le j \le 2p-1$ and is

$$\frac{p-1}{(j-1)(j-p)(\log n)^j} \quad \text{for } 2p \le j \le k+1.$$

Finally, deduce that $\sum_{n=1}^{\infty} \sigma_{k+1}(\delta_n)$ does not even converge.

The following auxiliary result will be fundamental.

Lemma 3.10: *Let (δ_n) be a G_p-convergent sequence of divisors in $\mathcal{V}$, for some $p \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} |h(\delta_n)|$ converges for every holomorphic function h in a neighbourhood of ∞ having a zero of order $\ge p$ at ∞.*

Proof: Let D be a V-disk such that h is holomorphic in D, and let $\sum_{j=p}^{\infty} \alpha_j z^j$ be the power series expansion of h in D with respect to z. Since $\overline{\lim}_{j\to\infty} \sqrt[j]{|\alpha_j|} < \infty$, there exists $R > 0$ such that $|\alpha_j| < R^j$ for every $j \in \mathbb{N}$. Now, by (3) in Proposition 3.2, $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n) R^j$ is G_p-convergent. Therefore, the double series $\sum_{j\ge p, n\ge n_0} |\sigma_j(\delta_n)| R^j$ converges for some $n_0 \in \mathbb{N}$, whence one deduces that $\sum_{j\ge p, n\ge n_0} |\alpha_j \sigma_j(\delta_n)|$ converges, and hence that $\sum_{n\ge n_0} |h(\delta_n)|$ converges. ∎

Theorem 3.11: *Let (δ_n) be a sequence of finite divisors in $\mathcal{V}$. Then, the following conditions are equivalent:*

(1) *There exists a sequence* (f_n) *in* G_∞ *having* $(\delta_n|_{\mathcal{V}'})$ *as sequence of divisors and such that* $\prod_{n=1}^\infty f_n$ *converges normally in* $\mathcal{V}'$.
(2) (δ_n) *is* G_{g+1}*-convergent.*
(3) (δ_n) *tends to* ∞ *and* $\sum_{n=1}^\infty |h(\delta_n)|$ *converges for every holomorphic function* h *in a neighbourhood of* ∞ *having a zero of order* $\geq g+1$ *at* ∞.

Proof: (1)⇒(2). Let $\epsilon > 0$, and D be a V-disk with radius $< \epsilon$. If (f_n) is as in the statement, then $\sum_{n=1}^\infty d\log f_n$ also converges normally in $\mathcal{V}'$. By Proposition 2.3 we can suppose that all the f_n are Δ-simple, and from (2) of Proposition 2.4 we deduce that $\int_{\partial D} z^j \, d\log f_n = \sigma_j(\delta_n)$ for every $n \in \mathbb{N}$ and $j \geq g+1$. Consider any such j, and let $n_0 \in \mathbb{N}$ be such that

$$\sum_{n=n_0}^\infty \left\| \frac{d\log f_n}{dz} \right\|_{\partial D} < 1,$$

with the usual notation denoting the supremum norm. Then, one has

$$\sum_{n=n_0}^\infty |\sigma_j(\delta_n| = \sum_{n=n_0}^\infty \left| \int_{\partial D} z^j \, d\log f_n \right| \leq 2\pi\epsilon^{j+1} \sum_{n=n_0}^\infty \left\| \frac{d\log f_n}{dz} \right\|_{\partial D} < 2\pi\epsilon^{j+1},$$

from which we easily deduce that $(\delta_n))$ is G_{g+1}-convergent.

(2)⇒(3). By Theorem 3.7 and Lemma 3.10.

(3)⇒(1). Let f_n be a Δ-simple normalized function in G_∞ with divisor $\delta_n|_{\mathcal{V}'}$, for every $n \in \mathbb{N}$. By Theorem 2.5 it will suffice to see that for every V-disk D and every holomorphic function ψ in some neighbourhood of $\overline{D}$ verifying $\psi(\infty) = 0$, the series $\sum_{n=1}^\infty \int_{\partial D} \psi \, d\log f_n$ converges absolutely. Let $\sum_{j=1}^\infty \alpha_j z^j$ be the power series expansion of ψ in some neighbourhood of $\overline{D}$ with respect to z, and let ψ_g be the function defined in this neighbourhood by $\psi_g = \sum_{j=g+1}^\infty \alpha_j z^j$. By (2) in Proposition 2.4, we have

$$\begin{aligned}\int_{\partial D} \psi \, d\log f_n &= 2\pi i \left(\psi(\delta_n) + \sum_{j=1}^g \alpha_j \varphi_j(\delta_n) \right) \\ &= 2\pi i \left(\psi_g(\delta_n) + \sum_{j=1}^g \alpha_j (\varphi_j(\delta_n) + \sigma_j(\delta_n)) \right),\end{aligned}$$

whence one obtains from Proposition 1.1 the desired conclusion. ∎

Corollary 3.12: *Let* (δ_n) *be a sequence of divisors in* $\mathcal{V}$. *Then the following conditions are equivalent:*

(1) $\sum_{n=1}^\infty \theta_{\delta_n \infty}$ *converges normally in* $\mathcal{V}'$.

(2) (δ_n) *is G-convergent.*

(3) (δ_n) *tends to* ∞ *and* $\sum_{n=1}^{\infty} |\psi(\delta_n)|$ *converges for every holomorphic function* ψ *in an open neighbourhood of* ∞*, having a zero at* ∞.

Proof: (1)⇒(2). Given $\epsilon > 0$, let D be a V-disk with radius $< \epsilon$. Then, (2) in Proposition 2.4 implies that $\int_{\partial D} z^j \theta_{\delta_n \infty} = \sigma_j(\delta_n)$ for every $j, n \in \mathbb{N}$, from which one deduces (2) as in the proof of Theorem 3.11.

(2)⇒(3). By Theorem 3.7 and Lemma 3.10.

(3)⇒(1). Let ψ be as in the statement. For every $n \in \mathbb{N}$ consider a Δ-simple normalized $f_n \in G_\infty$, with divisor δ_n. Then, by Theorem 3.11, $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, and so the same is true for $\sum_{n=1}^{\infty} d \log f_n$. Therefore, by Proposition 1.1 and (2) in Proposition 2.4, $\sum_{n=1}^{\infty} \theta_{\delta_n \infty}$ converges normally in $\mathcal{V}'$. ∎

Corollary 3.13: *Let (h_n) be a normalized sequence in $M(\mathcal{V})$ with sequence of divisors (δ_n). Then $\prod_{n=1}^{\infty} h_n$ converges normally in $\mathcal{V}'$ if and only if (δ_n) is G-convergent.*

Proof: Assume that $\prod_{n=1}^{\infty} h_n$ converges normally in $\mathcal{V}'$. Then, by Proposition 2.3 and (1) in Proposition 2.4, $\sum_{n=1}^{\infty} \theta_{\delta_n \infty}$ also does, which implies, by Corollary 3.12, that (δ_n) is G-convergent.

Conversely, if (δ_n) has this property, then again by Corollary 3.12, $\sum_{n=1}^{\infty} \theta_{\delta_n \infty}$ converges normally in $\mathcal{V}'$. In particular, $\lim_{n \to \infty} \int_{B_j} \theta_{\delta_n \infty} = 0$, for $1 \leq j \leq g$, whence one deduces (as in the proof of Theorem 2.17 in [3], for instance) that there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that h_n is Δ-simple, whereupon from (1) of Proposition 2.4 we obtain the desired conclusion. ∎

The following result includes as a particular case Theorem 3.11, and generalizes a previous one appearing in [3] for the bounded case. In its statement, for every $\ell \in \mathbb{N}$ with $\ell \geq g$, G_∞^ℓ denotes the group of meromorphic functions in $\mathcal{V}'$ having at ∞ a polynomic exponential singularity of degree $\leq \ell$ (and so G_∞^g is what we have previously called G_∞).

Theorem 3.14: *Let (δ_n) be a sequence of divisors in $\mathcal{V}$. Then, the following conditions are equivalent:*

(1) *There exists a sequence (f_n) in G_∞^ℓ having $(\delta_n|_{\mathcal{V}'})$ as sequence of divisors and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$.*

(2) (δ_n) *is $G_{\ell+1}$-convergent.*

(3) (δ_n) *tends to* ∞ *and* $\sum_{n=1}^{\infty} |h(\delta_n)|$ *converges for every holomorphic function* h *in an open neighbourhood of* ∞ *having a zero of order* $\geq \ell + 1$ *at this point.*

Proof: (1)⇒(2). As in the proof of Theorem 3.11.

(2)⇒(3). By Theorem 3.7 and Lemma 3.10.

(3)⇒(1). By Theorem 3.11 we can suppose that $\ell \geq g+1$. Reasoning as in the proof of Theorem 2.14 in [3], first choose, for every $n \in \mathbb{N}$, a Δ-simple and normalized $F_n \in G_\infty$ with divisor δ_n. Consider also $H_n \in M(\mathcal{V}) \cap O(\mathcal{V}')$, such that $\prod_{n=1}^{\infty} F_n e^{H_n}$ converges normally in $\mathcal{V}'$. Then use the Riemann–Roch theorem to obtain $h_n \in M(\mathcal{V}) \cap O(\mathcal{V}')$ such that the coefficients of $1/z^{g+1}, 1/z^{g+2}, \ldots, 1/z^{\ell}$ in its Laurent series in V' vanish and such that $\mathrm{ord}_\infty(H_n - h_n) \geq -\ell$. Finally, note that the proof finishes if $\sum_{n=1}^{\infty} h_n$ converges normally in $\mathcal{V}'$. To see this, argue as in the mentioned theorem. ∎

Note that in the proof of Theorem 3.14, condition (2) of its statement could have been replaced by the $G_{\ell+1}$-convergence of $\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$. Therefore, one deduces the validity of the observation made just before Theorem 3.5.

Remark: We need not investigate any characterization of the sequences (δ_n) as above such that there is a normally convergent product $\prod_{n=1}^{\infty} f_n$, with f_n belonging to some G_∞^ℓ (with ℓ possibly depending on n) and having δ_n as divisor for every $n \in \mathbb{N}$. In fact, there always exists such a product. This is Günther's theorem ([6]), of which a modern proof can be found in [3].

4. Sequences of positive divisors

In this section we shall deal with special properties of the sequences of positive divisors in $\mathcal{V}$, or existence theorems of certain sequences of such divisors. First, we see that G-convergent sequences (δ_n) of positive divisors in $\mathcal{V}$ can also be characterized by means of the double series $\sum_{j,n=1}^{\infty} \lambda_j(\delta_n)$ (recall that $\lambda_j(\delta_n) = 0$ for $j > \mathrm{degree}(\delta_n)$).

THEOREM 4.1: *Let* (δ_n) *be a sequence of positive divisors in* V. *Then,* (δ_n) *is* G*-convergent if and only if the double series* $\sum_{j,n=1}^{\infty} \lambda_j(\delta_n)$ *is* G*-convergent.*

Proof: Consider for any $n_0, N, j \in \mathbb{N}$, with $n_0 \leq N$, the equality

$$\sum_{n=n_0}^{N} \sigma_j(\delta_n) + \sum_{\ell=1}^{j-1} \sum_{n=n_0}^{N} \lambda_\ell(\delta_n)\sigma_{j-\ell}(\delta_n) + j \sum_{n=n_0}^{N} \lambda_j(\delta_n) = 0,$$

which is an obvious consequence of (2.5.1), and use it to prove by induction on j, as in the Remark of Section 2, that if $\sum_{n=n_0}^{\infty} |\sigma_j(\delta_n)| < \epsilon^j$ for some $\epsilon > 0$ and every $j \in \mathbb{N}$, then also $\sum_{n=n_0}^{\infty} |\lambda_j(\delta_n)| < \epsilon^j$ for every $j \in \mathbb{N}$, and also that if $\sum_{n=n_0}^{\infty} |\lambda_j(\delta_n)| < \epsilon^j$ for every $j \in \mathbb{N}$, then $\sum_{n=n_0}^{\infty} |\sigma_j(\delta_n)| < (2^j - 1)\epsilon^j$ for every $j \in \mathbb{N}$, from which we easily obtain the above conclusion. ∎

Theorem 4.1 cannot be generalized to the case of G_p-convergence, with $p \geq 2$. Indeed, if we consider any G-convergent sequence (δ_n) of positive divisors in $\mathcal{V}$ and a sequence (a_n) in V such that $\sum_{n=1}^{\infty} |z(a_n)|^p$ converges but $\sum_{n=1}^{\infty} |z(a_n)|^{p-1}$ does not, with $p \geq 3$, then $(\delta_n + a_n)$ is G_p-convergent (not G_{p-1}-convergent), but $\sum_{j,n=1}^{\infty} \lambda_j(\delta_n + a_n)$ is G_2-convergent.

The following proposition is very easy to prove. Simply take into account that $\sigma_1, \ldots, \sigma_k$ are coordinates in $V^{(k)}$.

PROPOSITION 4.2: *Let (δ_n) be a sequence of positive divisors in V with degrees bounded by some $k \in \mathbb{N}$. If $\sum_{n=1}^{\infty} |\sigma_j(\delta_n)|$ converges for $1 \leq j \leq k$, then (δ_n) tends to ∞.*

Note, however, that if we had instead considered a sequence (δ_n) of not necessarily positive divisors, then we could not have deduced from the only hypothesis of the absolute convergence of all its series of Newton sums that (δ_n) tends to ∞. Take for instance $\delta_n^+, \delta_n^- \in V^{(k)}$, for every $n \in \mathbb{N}$, whose coordinates $\sigma_1, \ldots, \sigma_k$ have moduli $\geq 1/2$, but verifying that $|\sigma_j(\delta_n^+) - \sigma_j(\delta_n^-)| < 1/2^n$, for $1 \leq j \leq k$ and every $n \in \mathbb{N}$ (which implies by Proposition 2.2 that $\sum_{n=1}^{\infty} |\sigma_j(\delta_n^+) - \sigma_j(\delta_n^-)|$ also converges, for every $j \in \mathbb{N}$, if we further suppose that both (δ_n^+), (δ_n^-) tend to ∞).

Relying on Proposition 4.2, one may perhaps expect that in the not necessarily bounded case the hypothesis that all the series $\sum_{n=1}^{\infty} |\sigma_j(\delta_n)|$ converge for $j \in \mathbb{N}$ could be sufficient to conclude that a sequence of positive divisors (δ_n) tends to ∞. We shall see that this is not true after proving two auxiliary results.

LEMMA 4.3: *For any $n \in \mathbb{N}$, and $(w_1, \ldots, w_n) \in \mathbb{C}^n$, with $|w_j| < 1/2^j$ for $1 \leq j \leq n$, there exists $\delta_n \in V^{(n)}$ such that $\sigma_j(\delta_n) = w_j$ for $1 \leq j \leq n$.*

Proof: By the remark following (2.5.1), for any divisor δ_n such that $|\sigma_j(\delta_n)| < 1/2^j$ for $1 \leq j \leq n$, one also has $|\lambda_j(\delta_n)| < 1/2^j$ for $1 \leq j \leq n$. Therefore, the desired conclusion can be obtained from a well-known inequality concerning polynomials (see, for instance, Proposition 1.1 in [8]). ∎

In the following lemma and theorem, we shall consider a V-disk W with radius $1/3$.

LEMMA 4.4: *For every $n \in \mathbb{N}$ and $\epsilon > 0$ there exists $\delta_n \in V^{(n)} - W^{(n)}$ such that $|\sigma_j(\delta_n)| < \epsilon^j$ for $1 \leq j \leq n-1$.*

Proof: We can of course suppose that $\epsilon < 1/2$. From Newton's formula, $\sigma_n(\delta_n) + \lambda_1(\delta_n)\sigma_{n-1}(\delta_n) + \cdots + \lambda_{n-1}(\delta_n)\sigma_1(\delta_n) + n\lambda_n(\delta_n) = 0$, with $\delta_n \in V^{(n)}$, one obtains an equality of the type $\sigma_n(\delta_n) + P_n(\sigma_1(\delta_n), \ldots, \sigma_{n-1}(\delta_n)) + n\lambda_n(\delta_n) = 0$, for some polynomial P_n in $n-1$ variables. Since the polynomial P_n is not identically zero, we can consider, for any $n \in \mathbb{N}$, $w_1, \ldots, w_{n-1} \in \mathbb{C}$ such that $|w_j| < \epsilon^j$, for $1 \leq j \leq n-1$, and $P_n(w_1, \ldots, w_{n-1}) \neq 0$. Let $w_n \in \mathbb{C}$, with $|w_n| < 1/2^n$, be such that $|w_n + P_n(w_1, \ldots, w_{n-1})| > 1/2^n$. Then, by Lemma 4.3, there exists $\delta_n = \sum_{i=1}^{n} a_{n,i} \in V^{(n)}$ such that $\sigma_j(\delta_n) = w_j$ for $1 \leq j \leq n$. Taking now into account that $w_n + P_n(w_1, \ldots, w_{n-1}) = -n\lambda_n(\delta_n)$, we see that $|\lambda_n(\delta_n)| > 1/n2^n$, which implies that at least one of the $|z(a_{n,i})|$ is $> \sqrt[n]{1/n2^n} > 1/3$, that is, at least one of the $a_{n,i}$ does not belong to W, as required. ∎

It is already possible to prove in an easy way the above mentioned:

THEOREM 4.5: *There exists a sequence (δ_n) of positive divisors in $\mathcal{V}$ such that $\sum_{n=1}^{\infty} |\sigma_j(\delta_n)|$ converges for every $j \in \mathbb{N}$, but (δ_n) does not tend to ∞.*

Proof: Consider for every $n \in \mathbb{N}$, $\delta_n \in V^{(n)} - W^{(n)}$, such that $|\sigma_j(\delta_n)| < 1/2^{jn}$ for $1 \leq j \leq n-1$ and $n \geq 2$. ∎

We finish this section by showing the existence of sequences (δ_n) of positive finite divisors in $\mathcal{V}'$, such that:

(1) The series of the j-th Newton sums of the δ_n converges absolutely for every $j \in \mathbb{N}$.

(2) (δ_n) tends to ∞.

(3) (δ_n) is not G-convergent.

Note that condition (2) is not necessarily a consequence of (1) by Theorem 4.5. Note also that it is easy to see that there are series $\sum_{j,n=1}^{\infty} \alpha_{j,n}$ with all its rows absolutely convergent but not G-convergent (take, for instance, $\alpha_{j,n} = (j/j+1)^n$, for $j, n \in \mathbb{N}$). However, this is not sufficient to obtain the above conclusion, since for a positive finite divisor $\delta \in V^{(n)}$, with $n \in \mathbb{N}$, it is not possible to arbitrarily preassign the values of all the $\sigma_j(\delta)$ but only the first n ones. Furthermore, the values of the $\sigma_j(\delta_n)$ that we can arbitrarily prescribe must guarantee that (δ_n) tends to ∞, or we can alternatively choose the points defining the divisors δ_n in a suitable way, as in the following:

THEOREM 4.6: *Let (α_n) be any sequence of real numbers of the interval $(0,1)$ converging to 0, ρ_n be a primitive n-th root of unity for every $n \in \mathbb{N}$, and δ'_n be the divisor $\sum_{i=1}^{n} a_{n,i}$, with $a_{n,i} \in V$ and $z(a_{n,i}) = \alpha_n \rho_n^i$, for $1 \leq i \leq n$. Then, there exists a sequence (k_n) of natural numbers such that the sequence (δ_n), defined by $\delta_n = \delta'_m$ for each $n \in \mathbb{N}$, where $m \in \mathbb{N}$ is minimum verifying that $n \leq k_1 + \cdots + k_m$, fulfils the above three properties.*

Proof: First, note that (δ'_n) tends to ∞ and that, for $j \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} \sigma_j(\delta'_n)$ are absolutely convergent because all but a finite set of their terms are zero. Therefore, the same must be true for any sequence of the type of the (δ_n) as in the statement.

Let $F_n \in G_\infty$ be Δ-simple, normalized and with divisor δ'_n, for every $n \in \mathbb{N}$. Then, it is easy to see that there is a sequence (k_n) of natural numbers such that $\prod_{n=1}^{\infty} F_n^{k_n}$ does not converge normally in $\mathcal{V}'$. Consider now the sequence (f_n) whose first k_1 terms are equal to F_1, followed by k_2 terms equal to F_2, and so on (i.e., $f_n = F_m$, with $m \in \mathbb{N}$ minimum such that $n \leq k_1 + \cdots + k_m$, for every $n \in \mathbb{N}$). Then, it is clear that $\prod_{n=1}^{\infty} f_n$ does not converge normally in $\mathcal{V}'$, whence by Proposition 2.3 and Theorem 3.11 one obtains the desired conclusion. ∎

5. Generalizations

First, we want to show that some of the foregoing results are valid, with the natural modifications, in the more general case in which ∞ may be a Weierstrass point. Unless otherwise stated we maintain the notations used in the preceding sections.

Let $\ell_1 < \cdots < \ell_g$ be the Weierstrass gaps at ∞, and $L = \{\ell_1, \ldots, \ell_g\}$.

Definition 5.1: A holomorphic function in any V-disk, taking the value 0 at ∞ and having null coefficients of $z^{\ell_1}, \ldots, z^{\ell_g}$ in its Taylor series, will be called a L-function.

For $j = 1, \ldots, g$, let θ_j be the unique holomorphic differential in $\mathcal{V}'$ such that $\theta_j - dz/z^{\ell_j+1}$ is holomorphic in V, and such that $\int_{A_i} \theta_j = 0$ for every $i = 1, \ldots, g$. For every $a \in \Delta'$ and $j = 1, \ldots, g$, let $\varphi_j(a) \in \mathbb{C}$ be such that $\theta_{a\infty} + \sum_{j=1}^{g} \varphi_j(a)\theta_j$ has null integrals along $B_1, \ldots, B_g$. Then, as in Proposition 1.1, $\varphi_1, \ldots, \varphi_g$ prolong to holomorphic functions in Δ, and each sum $\varphi_j + z^{\ell_j}$, with $1 \leq j \leq g$, is a L-function. For any $a \in \Delta' - \{q_0\}$, let $f_a \in G(\mathcal{V}')$ be the unique normalized function defined by $d\log f_a = \theta_{a\infty} + \sum_{j=1}^{g} \varphi_j(a)\theta_j$, and consider the multiplicative group G_∞ generated over $\mathcal{M}^*(\mathcal{V})$ $(= M(\mathcal{V}) - \{0\})$

by all these functions f_a. Then, G_∞ is the subgroup of $G(\mathcal{V}')$ formed by the functions which are in V' of the form $he^{P(1/z)}$, with h meromorphic in V and P a linear combination with coefficients in $\mathbb{C}$ of $1/z^{\ell_1}, \dots, 1/z^{\ell_g}$ (see [2]). So this definition coincides with the previous one when ∞ is not of Weierstrass and, as in that case, G_∞ contains functions with all possible finite divisors in $\mathcal{V}'$.

Definition 5.2: A double series $\sum_{j,n=1}^{\infty} \alpha_{j,n}$, with $\alpha_{j,n} \in \mathbb{C}$ for all $j, n \in \mathbb{N}$, will be called L-geometrically convergent (or G_L-convergent for short) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\alpha_{j,n}| < \epsilon^j$, for every $j \in \mathbb{N} - L$.

This type of series corresponds to normally convergent products of functions in G_∞ in a similar way as in Theorem 3.11. Note that in this more general case, Theorem 3.5 (with the same statement) remains valid. However, in relation to a possible definition of G_L-convergence, it is not possible to use Lemma 3.4 to deduce directly an analogue of Theorem 3.5 (save for the case of all the functions being meromorphic in $\mathcal{V}$, as is not difficult to see using the above-mentioned extension of Proposition 1.1). Anyway, as a consequence of the following generalization of Theorem 3.11, the condition in Definition 5.2, applied to a series of the type $\sum_{j\in\mathbb{N}, n\geq N} \sigma_j(\delta_n)$, is independent of the coordinate z considered. Thus, it makes sense to speak of G_L-convergent sequences of divisors in $\mathcal{V}$ in the analogous sense to that of Definition 3.8. Note that, by Theorem 3.7, such sequences must tend to ∞.

Theorem 5.3: *Let (δ_n) be a sequence of divisors in $\mathcal{V}$. Then, the following conditions are equivalent:*

1. *There exists a sequence (f_n) in G_∞ having $(\delta_n|_{\mathcal{V}'})$ as sequence of divisors and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$.*
2. *$\sum_{j,n=1}^{\infty} \sigma_j(\delta_n)$ is G_L-convergent.*
3. *(δ_n) tends to ∞ and $\sum_{n=1}^{\infty} |h(\delta_n)|$ converges for every L-function h.*

Proof: Similar to that of Theorem 3.11. ∎

From Theorem 5.3 one can obtain an analogue of Corollary 3.12 and also the following:

Corollary 5.4: *Let (h_n) be a normalized sequence in $M(\mathcal{V})$ with sequence of divisors (δ_n). Then $\prod_{n=1}^{\infty} h_n$ converges normally in $\mathcal{V}'$ if and only if (δ_n) is G-convergent.*

We now undertake a second brief generalization of part of what we have exposed in previous sections. From now on $\mathcal{V}''$ will be the complementary in

$\mathcal{V}$ of a finite subset $S = \{\infty_1, \ldots, \infty_r\}$, with $r \geq 2$. We shall also consider for $1 \leq i \leq r$ a coordinate disk V_i in $\mathcal{V}$ centered at ∞_i, such that $\overline{V}_i \subset \Delta$, with $\overline{V}_1, \ldots, \overline{V}_r$ disjoint, and such that the point q_0 (which we are using to normalize functions) belongs to $\mathcal{V} - \bigcup_{i=1}^n \overline{V}_i$. V_i' will be $V_i - \{\infty_i\}$. L_i will be the set of Weierstrass gaps at ∞_i.

We shall put for simplicity $\infty = \infty_1$, $\mathcal{V}' = \mathcal{V} - \{\infty_1\}$, $V = V_1$ and $L = L_1$. As above, $l_1 < \cdots < l_g$ will be the Weierstrass gaps at ∞. When it does make sense and there is no danger of confusion, we shall continue using the terminology previously employed for the point ∞, and the analogues for the rest of points of S (for instance, the expressions V_i-disk, G_{L_i}-convergence, etc).

Definition 5.5: A holomorphic function in any neighbourhood of S which is a L-function (at ∞) and takes the value 0 at all points of S will be called a $\mathcal{L}$-function.

Definition 5.6: We shall say that a sequence (δ_n) of divisors in $\mathcal{V}$ is $G_{\mathcal{L}}$-convergent (resp. G-convergent) to S if the following conditions hold:

(1) All but a finite set of the δ_n are supported in $\bigcup_{i=1}^n V_i$.
(2) $(\delta_n|_V)$ is G_L-convergent (resp. G-convergent) to ∞.
(3) $(\delta_n|_{V_i})$ is G-convergent to ∞_i, for $i = 2, \ldots, r$.

Note that, by Theorem 3.7, if $(\delta_n|_{V_i})$ is G_{L_i}-convergent to ∞_i, then the sequence $(\delta_n|_{V_i})$ also tends to ∞_i. So, if (δ_n) is $G_{\mathcal{L}}$-convergent to S, it tends to S too (in the obvious sense that every neighbourhood of S contains all but finitely many supports of these divisors). Note also that it may happen that a sequence of divisors (δ_n) in $\mathcal{V}$ tends to S and one (or more) of the V_i contains no point of the supports of the δ_n. Something similar is of course possible in relation to the $G_{\mathcal{L}}$-convergence.

THEOREM 5.7: *Let (f_n) be a normalized sequence in G_∞ whose sequence of divisors (δ_n) is G-convergent to a point ∞' of $\mathcal{V}'$. Then, $\prod_{n=1}^\infty f_n$ converges normally in $\mathcal{V}' - \{\infty'\}$.*

Proof: Let $G_{\infty'}$ be the analogue of G_∞ corresponding to ∞'. Let E be the subspace of $O(\mathcal{V}' - \{\infty'\})$ formed by the functions taking at q_0 the value 0, having at ∞ a singular part of the type

$$\frac{\alpha_{-(\ell_g+1)}}{z^{\ell_g+1}} + \cdots + \frac{\alpha_{-(\ell_1+1)}}{z^{\ell_1+1}},$$

with $\alpha_{-(\ell_g+1)}, \ldots, \alpha_{-(\ell_1+1)} \in \mathbb{C}$, and with a singular part at ∞' of the same type

$$\frac{\alpha'_{-(\ell'_g+1)}}{z'^{\ell'_g+1}} + \cdots + \frac{\alpha'_{-(\ell'_1+1)}}{z'^{\ell'_1+1}},$$

where z' is a coordinate, vanishing at ∞', in some disk D centered at ∞', $\ell'_1, \ldots, \ell'_g$ are the Weierstrass gaps at ∞' and $\alpha'_{-(\ell_g+1)}, \ldots, \alpha'_{-(\ell_1+1)} \in \mathbb{C}$. For every $n \in \mathbb{N}$, let H_n be the unique function in E such that $f_n e^{H_n} \in G_{\infty'}$ (which exists by the Riemann–Roch theorem, for instance). Then, by Theorem 5.3, $\prod_{n=1}^{\infty} f_n e^{H_n}$ converges normally in $\mathcal{V} - \{\infty'\}$.

From Proposition 2.3 and the analogue of (2) in Proposition 2.4, we see that there exists $n_0 \in \mathbb{N}$ such that the coefficient of $1/z'^{\ell'_j}$ in the singular part of H_n at ∞' can also be expressed as $\varphi'_j(\delta_n)/\ell'_j z'^{\ell'_j}$, for $n \geq n_0$ and $1 \leq j \leq g$, where $\varphi'_1, \ldots, \varphi'_g$ are the analogues for the point ∞' to the $\varphi_1, \ldots, \varphi_g$ above considered. Then, since the mapping from E into $\mathbb{C}^g$ defined by $h \to (\alpha'_{-(\ell'_g+1)}, \ldots, \alpha'_{-(\ell'_1+1)})$ (notations as above) is an isomorphism of $\mathbb{C}$-vector spaces, we deduce from the G-convergence of (δ_n) to ∞' that, for $1 \leq j \leq g$, the series $\sum_{n=1}^{\infty} |\varphi'_j(\delta_n)|$ converges, that is, the series of coordinates in E of the functions H_n converge absolutely. Therefore, the series $\sum_{n=1}^{\infty} H_n$ converges normally in $\mathcal{V}' - \{\infty'\}$, from which it clearly results that $\prod_{n=1}^{\infty} f_n$ also converges normally in $\mathcal{V}' - \{\infty'\}$. ∎

THEOREM 5.8: *Let (δ_n) be a sequence of divisors in $\mathcal{V}$. Then, the following conditions are equivalent:*

(1) *There exists a sequence (f_n) in G_∞ having $(\delta_n|_{\mathcal{V}'})$ as sequence of divisors and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}''$.*
(2) *(δ_n) is $G_{\mathcal{L}}$-convergent to S.*
(3) *(δ_n) tends to S and $\sum_{n=1}^{\infty} |h(\delta_n)|$ converges for every $\mathcal{L}$-function h.*

Proof: (1)⇒(2). Let (f_n) be as in condition (1). Then, each product $\prod_{n=1}^{\infty} f_n|_{V'_i}$ converges normally in V'_i too, for $i = 1, \ldots, r$, and one deduces as in the proof of Theorem 3.11 that (δ_n) is G_L-convergent to ∞ and G-convergent to every other point of S. Thus, (δ_n) is $G_{\mathcal{L}}$-convergent to S.

(2)⇒(3). Let ψ be a $\mathcal{L}$-function, and consider, for $i = 1, \ldots, r$, a V_i-disk D_i such that ψ is holomorphic in $\bigcup_{i=1}^{n} D_i$. Let ψ_i be the restriction of ψ to D_i. Then, one sees as in the proof of Lemma 3.10 that $\sum_{n=1}^{\infty} |\psi_i(\delta_n|_{D_i})|$ converges for $i = 1, \ldots, r$, and so $\sum_{n=1}^{\infty} |\psi(\delta_n)|$ converges too.

(3)⇒(1). We can suppose that δ_n is supported in $\bigcup_{i=1}^{n} V_i$, for every $n \in \mathbb{N}$. Let $\delta_{i,n}$ be $\delta_n|_{V_i}$, for $1 \leq i \leq r$ and $n \in \mathbb{N}$. Then it is easy to see that $(\delta_{1,n})$

verifies condition (3) of Theorem 5.3, and so this theorem implies that $(\delta_{1,n})$ is G_L-convergent to ∞. Similarly, one sees that for $2 \leq i \leq r$, $(\delta_{i,n})$ verifies the analogue, for the point ∞_i, of (3) in Corollary 3.12, whence it results that $(\delta_{i,n})$ is G-convergent to ∞_i, for $2 \leq i \leq r$.

Let $f_{i,n} \in G_\infty$ be Δ-simple, normalized and with $\delta_{i,n}$ as divisor, for $1 \leq i \leq r$ and $n \in \mathbb{N}$. Then, by Theorem 5.3, $\prod_{n=1}^{\infty} f_{1,n}$ converges normally in $\mathcal{V}'$, and by Theorem 5.7, $\prod_{n=1}^{\infty} f_{i,n}$ converges normally in $\mathcal{V}' - \{\infty_i\}$ for $2 \leq i \leq r$. Let f_n be $\prod_{i=1}^{r} f_{i,n}$ for every $n \in \mathbb{N}$. Then $f_n \in G_\infty$, its divisor is $\delta_n|_{\mathcal{V}'}$ for every $n \in \mathbb{N}$, and $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}''$. ∎

COROLLARY 5.9: *Let (h_n) be a normalized sequence in $M(\mathcal{V})$ and (δ_n) be its sequence of divisors. Then $\prod_{n=1}^{\infty} h_n$ converge normally in $\mathcal{V}'$ if and only if (δ_n) is G-convergent to S.*

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